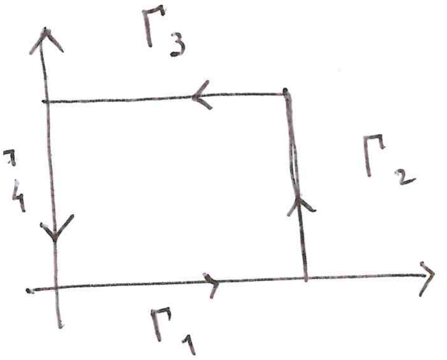
ES 1

$$\underline{f} = (3x+2y)\,\hat{x} + (x-y)\,\hat{y}$$

$$\omega(x,y) = (3x+2y)\,dx + (x-y)\,dy$$



$$\Gamma_i : \begin{cases} x = f(t) \\ y = h(t) \end{cases} \quad t \in [t_0, t_1]$$

$$\Gamma_i \cap \Gamma_j = \phi \quad \bigcup_{i=1}^{4} \Gamma_i = \partial A$$
$$i \neq j$$

$$\Gamma_1 : \begin{cases} x = t \\ y = 0 \end{cases} \quad t \in [0,1] \quad \int_{\Gamma_1} \omega_1 = \int_0^1 3t\,dt = \frac{3}{2}$$

$$\Gamma_2 : \begin{cases} x = 1 \\ y = t \end{cases} \quad t \in [0,1] \quad \int_{\Gamma_2} \omega_2 = \int_0^1 1-t\,dt = \frac{1}{2}$$

$$\Gamma_3 : \begin{cases} x = t \\ y = 1 \end{cases} \quad t \in [1,0] \quad \int_{\Gamma_3} \omega_3 = -\int_0^1 (3t+2)\,dt = -\frac{7}{2}$$

$$\Gamma_4 : \begin{cases} x = 0 \\ y = t \end{cases} \quad t \in [1,0] \quad \int_{\Gamma_4} \omega_4 = +\int_0^1 t\,dt = \frac{1}{2}$$

$$\oint_{\Gamma} \omega = \oint_{\Gamma} f \cdot \tau\,ds = \sum_{i=1}^{4} \int_{\Gamma_i} \omega_i = -\frac{1}{2} \cdot 2 = -1$$

$\psi = 3x^2 - 2y^3$ **ES 2**

$\psi(x,y,t) \;|\; \dfrac{\partial \psi}{\partial y} = u \qquad \dfrac{\partial \psi}{\partial x} = -v$

$\underline{\psi} = \psi \hat{k} \quad \Rightarrow \nabla \times \underline{\psi} = u \quad \Rightarrow \nabla \cdot u = 0$

ψ è un diff esatto $\Rightarrow d\psi = -v\,dx + u\,dy$

$u = -6y^2 \qquad v = -6x \qquad \boxed{\nabla \cdot u = 0} \rightarrow$

Il flusso è incompressibile

$$\nabla \times u = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & 0 \\ -6y^2 & -6x & 0 \end{vmatrix} = \hat{k}\,(-6 + 12y)$$

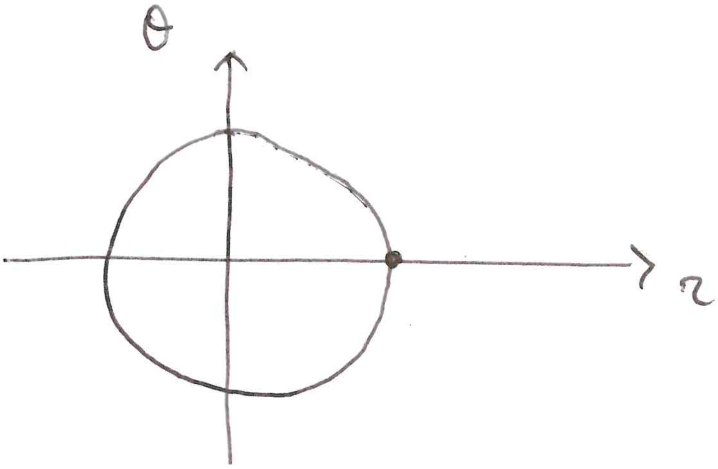
$$(\nabla \times u)_k = 12y - 6 = -\nabla^2 \psi = -(6 - 12y)$$

Non è irrotazionale!

ES3

$$u_1 = -\frac{1}{2}\,\frac{x_2}{x_1^2 + x_2^2}$$

$$u_2 = \frac{1}{2}\,\frac{x_1}{x_1^2 + x_2^2}$$



$$x_1 = r\cos\theta \qquad r \in \mathbb{R}^+$$

$$y_1 = r\sin\theta \qquad \theta \in [0, 2\pi]$$

$$= (\cos\theta, \sin\theta) \qquad \hat{\theta} = (-\sin\theta, \cos\theta)$$

$$u_1(r,\theta) = -\frac{1}{2}\,\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{2r}$$

$$u_2(r,\theta) = \frac{1}{2}\,\frac{r\cos\theta}{r^2} = \frac{\cos\theta}{2r}$$

$$u_r(r,\theta) = u \cdot \hat{r} = 0 \qquad u_\theta(r,\theta) = u \cdot \hat{\theta} = \frac{1}{2r}$$

$$u = \frac{1}{2r}\,\hat{\theta}$$ VORTICE CONCENTRATO

$$\Gamma = \int_0^{2\pi} u_\theta R \, d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi$$

$$\Gamma = \oint_\ell u \cdot \tau \, ds = \int_\sigma (\nabla \times u) \cdot n \, dS$$

$$\nabla \times u = \bar{\Gamma}\,\delta(x - x_0)\,\hat{k} \qquad \bar{\Gamma} = \frac{1}{2}$$

$$ = 2\pi \int \frac{1}{2}\, \delta(x - x_0)\, dx = \pi $$

ES 4

$\phi = x^2 - y^2 \qquad u = \nabla \phi$

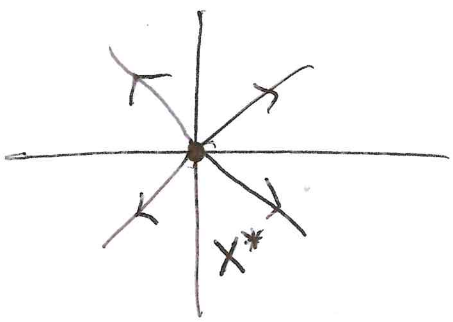
$u_x = \dfrac{\partial \phi}{\partial x} = 2x \qquad u_y = -2y$

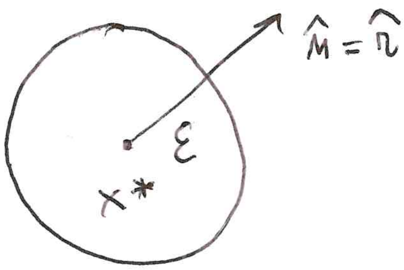
Due modi $\quad 1 \Rightarrow$ FORMA ESATTA $\Rightarrow \nabla \times u = 0$

$$\Rightarrow \nabla \times u = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & 0 \\ 2x & -2y & 0 \end{vmatrix} = 0\, \hat{k}$$

ES 5

SORGENTE UNITARIA





$B_\varepsilon(x^*)$

$$\nabla_x^2 g(x, x^*) = \delta_\varepsilon(x - x^*)$$

$$\delta_\varepsilon = \begin{cases} 0 & x \in \mathbb{R}^3 \setminus B_\varepsilon(x^*) \\ \lim_{\varepsilon \to 0} \delta_\varepsilon & x \in B_\varepsilon(x^* \end{cases}$$

$$\frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial g}{\partial r}\right) = 0 \qquad \text{fuori da} \quad B_\varepsilon(x^*)$$

$$r^2 \frac{\partial g}{\partial r} = A \quad \Rightarrow \quad g(r) = -\frac{A}{r} + \cancel{B}$$

$$\nabla g = \frac{A}{r^2}\hat{r}$$

$$\in B_\varepsilon \qquad \int_{B_\varepsilon} \nabla^2 g \, dV = \lim_{\varepsilon \to 0} \int_{B_\varepsilon} \delta_\varepsilon(x - x^*) \, dV = 1$$

$$\int_{\partial B_\varepsilon} \nabla g \cdot n \, dS = \frac{A}{\cancel{\varepsilon^2}} 4\pi \cancel{\varepsilon^2} = 4\pi A$$

$$\Rightarrow \quad A = \frac{1}{4\pi} \quad \Rightarrow \quad g(x) = -\frac{1}{4\pi |x|}$$

Potenziale elettrico carica puntiforme

uno uniforme $\quad U = U_0 \hat{x}$

$$= U_0 x$$

$$_{TOT} = g(x) + \Phi$$