ES 1

Si ricorda che: $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ $\quad \hat{e}_i \times \hat{e}_j = \varepsilon_{ijk}\hat{e}_k$

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})$$

$$+ \delta_{im}(\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})$$

e quindi

$$\varepsilon_{ijk}\varepsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

1-1*

$$(a \times (b \times c))_i = \varepsilon_{ijk} a_j (b \times c)_k$$

$$= \varepsilon_{ijk}\varepsilon_{kqp} a_j b_q c_p = \delta_{iq}\delta_{jp} a_j b_q c_p +$$

$$- \delta_{ip}\delta_{jp} a_j b_q c_p = a_j b_i c_j - a_j b_j c_i$$

$$\Rightarrow \quad a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c$$

1-2*

$$[\nabla \times (\nabla \times u)]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times u)_k$$

$$= \varepsilon_{ijk}\,\varepsilon_{kpq}\,\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_p}u_q = (\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp})\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_p}u_q$$

$$= \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_i} - \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j}$$

$$\Rightarrow \nabla\times(\nabla\times u) = \nabla(\nabla\cdot u) - \nabla^2 u$$

1-3*

$$\left[\nabla\times(f u)\right]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j} f u_k =$$

$$= \varepsilon_{ijk}\frac{\partial f}{\partial x_j}u_k + \varepsilon_{ijk} f \frac{\partial u_k}{\partial x_j}$$

$$\Rightarrow \nabla\times f u = \nabla f\times u + f\,\nabla\times u$$

1-4*

Chiamo $A_i = (\nabla\times\nabla f)_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} = \varepsilon_{ijk} H_{jk}$

H_{jk} Matrice Hessiana $H_{jk} = H_{kj}$ (TEOREMA SCHWARTZ

$$A_i = \varepsilon_{ijk} H_{kj} = -\varepsilon_{ikj} H_{kj} = -\varepsilon_{ijk} H_{jk} = -A_i$$

$$\Rightarrow A_i = -A_i \Rightarrow A_i = 0$$

NB: SE IL SIMBOLO DI LEVI CIVITA AGISCE SU OPERANDI TRASPONIBILI IL RISULTATO DELL'OPERAZIONE SARÀ ZERO

Cioè Se $L_{\alpha\beta} = L_{\beta\alpha}$

$$\varepsilon_{\gamma\alpha\beta} L_{\alpha\beta} = 0$$

Vale anche il viceversa.

1-5*

$$[\nabla \cdot (\nabla \times u)]_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j \partial x_i} = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_i \partial x_j} =$$

$$= -\varepsilon_{jik} \frac{\partial u_k}{\partial x_i \partial x_j} = -\varepsilon_{ijk} \frac{\partial u_k}{\partial x_j \partial x_i}$$

$$\Rightarrow \nabla \cdot (\nabla \times u) = 0$$

1-6*

$$[u \times (\nabla \times u)]_i = \varepsilon_{ijk} u_j (\nabla \times u)_k =$$

$$= \varepsilon_{ijk} \varepsilon_{kqp} u_j \frac{\partial}{\partial x_q} u_p = (\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jp}) u_j \frac{\partial u_p}{\partial x_q} =$$

$$= u_j \frac{\partial u_j}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_i}(u_j u_j / 2) - u_j \frac{\partial u_i}{\partial x_j}$$

$$\Rightarrow \quad u \times (\nabla \times u) = \nabla\left(\frac{u \cdot u}{2}\right) - u \cdot \nabla u$$

$1-7^*$

$$\nabla \cdot \nabla f = \frac{\partial}{\partial x_i}\hat{e}_i \cdot \frac{\partial f}{\partial x_j}\hat{e}_j = \frac{\partial^2 f}{\partial x_i^2}$$

$1-8^*$

$$\nabla \cdot (f u) = \frac{\partial}{\partial x_i}\hat{e}_i \cdot f u_j \hat{e}_j = \frac{\partial f}{\partial x_i} u_i + f \frac{\partial u_i}{\partial x_i}$$

$$= \nabla f \cdot u + f \nabla \cdot u$$

ES2

$$\begin{cases} u_1 = 2x_1 + x_3 \\ u_2 = x_1 - x_2 \\ u_3 = 3x_1 + 2x_2 \end{cases}$$

$$u \cdot u = 4x_1^2 + x_3^2 + 4x_1x_3 + x_1^2 + x_2^2 - 2x_1x_2 + 9x_1^2 + 4x_2^2$$

$$+ 12x_1x_2 = 14x_1^2 + 5x_2^2 + x_3^2 + 10x_1x_2 + 4x_1x_3$$

ES 3

$$\nabla \cdot u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 2 - 1 = 1$$

$$\nabla\left(\frac{u \cdot u}{2}\right) = \frac{1}{2}\hat{e}_1\left(28x_1 + 10x_2 + 4x_3\right) + \frac{1}{2}\hat{e}_2\left(10x_2 + 10x_1\right)$$

$$+ \frac{1}{2}\hat{e}_3\left(2x_3 + 4x_1\right)$$

$$(\nabla \times u)_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$

$$(\nabla \times u)_1 = \varepsilon_{123}\frac{\partial u_3}{\partial x_1} + \varepsilon_{132}\frac{\partial u_1}{\partial x_3} = 2$$

$$(\nabla \times u)_2 = \varepsilon_{231}\frac{\partial u_1}{\partial x_3} + \varepsilon_{213}\frac{\partial u_3}{\partial x_1} = -2$$

$$(\nabla \times u)_3 = \varepsilon_{312}\frac{\partial u_2}{\partial x_1} + \varepsilon_{321}\frac{\partial u_1}{\partial x_2} = 1$$

S4

$$u_1 = 2x_1 - x_2 + t$$

$$u_2 = 4x_1 + 1$$

$$u_3 = 0$$

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$$

$$1: \quad \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial x_3} =$$

$$: \quad 1 + 2(2x_1 - x_2 + t) + (-1)(4x_1 + 1)$$

$$= \cancel{1} + \cancel{4x_1} - 2x_2 + 2t - \cancel{4x_1} - \cancel{1} = 2(t - x_2)$$

$$2: \quad \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_3}{\partial x_3} =$$

$$= 4(2x_1 - x_2 + t)$$

$$3: \quad \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x_1} + u_2 \frac{\partial u_3}{\partial x_2} + u_3 \frac{\partial u_3}{\partial x_3} = 0$$

S5

$$\frac{Du_i}{Dt} = \rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial T_{ij}}{\partial x_j},$$

$$1: \ \rho\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_1}{\partial x_3}\right) = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}$$

$$2: \ \rho\left(\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + u_3\frac{\partial u_2}{\partial x_3}\right) = \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3}$$

$$3: \ \rho\left(\frac{\partial u_3}{\partial t} + u_1\frac{\partial u_3}{\partial x_1} + u_2\frac{\partial u_3}{\partial x_2} + u_3\frac{\partial u_3}{\partial x_3}\right) = \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3}$$

In 2 dimensioni abbiamo :

$$1: \ \rho\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2}\right) = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2}$$

$$2: \ \rho\left(\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2}\right) = \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2}$$

ES 6

forma conservativa

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial T_{ij}}{\partial x_j}$$

$$u_i\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_j}u_j + \rho\frac{\partial u_j}{\partial x_j}\right) + \rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = \frac{\partial T_{ij}}{\partial x_j}$$

$$\frac{\partial \rho}{\partial t} + u\cdot\nabla\rho + \rho\,\nabla\cdot u = \frac{D\rho}{Dt} + \rho\,\nabla\cdot u = 0$$