ESERCITAZIONE CINEMATICA (SOLUZIONI)

Mappa di moto: $x = \chi_k(X, t)$

$$\chi_k : \begin{cases} x_1 = X_1 + t X_2 \\ x_2 = X_2 + t X_1 \end{cases}$$

ES 1-1*

$$u = \left.\frac{\partial \chi_k}{\partial t}\right|_X \Rightarrow u : \begin{cases} u_1 = X_2 \\ u_2 = X_1 \end{cases}$$

ES 1-2*

Mappa di moto inversa

$$x = \chi_k(X, t) \Longrightarrow X = \chi_k^{-1}(x, t)$$

$$\begin{cases} X_1 = \dfrac{x_1 - t x_2}{1 - t^2} \\ X_2 = \dfrac{x_2 - t x_1}{1 - t^2} \end{cases}$$

ES1-3*

Rappresentazione Euleriana di u

$$u = \left.\frac{\partial \chi_\kappa}{\partial t}\right|_X \Rightarrow u = u(X, t) = u(\chi_\kappa^{-1}(x,t), t)$$

$$\Rightarrow u = u(x, t)$$

$$\begin{cases} u_1 = X_2 \\ u_2 = X_1 \end{cases} \xrightarrow{\chi_\kappa^{-1}} \begin{cases} u_1 = \dfrac{x_2 - t x_1}{1 - t^2} \\ u_2 = \dfrac{x_1 - t x_2}{1 - t^2} \end{cases}$$

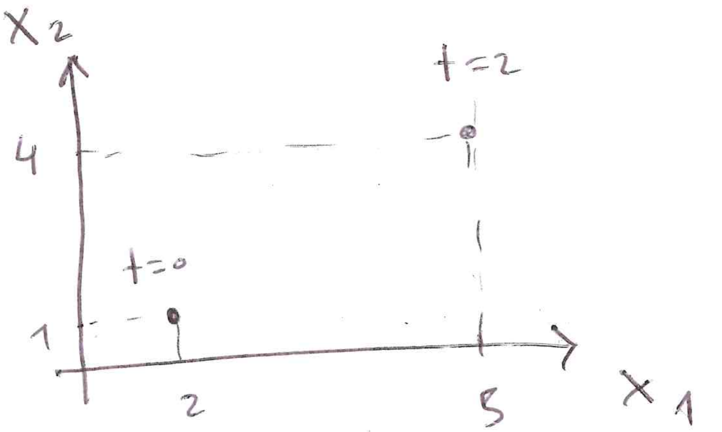
ES1-4*

Velocità Lagrangiana al tempo $t = 2$ della particella che all'istante iniziale si trova in $X = (2, 1)$

$$\begin{cases} u_1 = 1 \\ u_2 = 2 \end{cases}$$

ES 1-5*

Calcolare della posizione, e delle velocità Euleriane



$$\begin{cases} x_1 = 2 + t \\ x_2 = 1 + 2t \end{cases}$$

$$\text{in } t = 2 \Rightarrow \begin{cases} x_1 = 4 \\ x_2 = 5 \end{cases}$$

$$d_{(t=0,\, t=2)} = 2\sqrt{5} \Rightarrow \langle v \rangle = u \cdot u = \frac{d(t=0,\, t=2)}{2}$$

$$= 5;$$

La velocità Euleriane sono:

$$\begin{cases} u_1 = \dfrac{5-8}{-3} = 1 \\[2ex] u_2 = \dfrac{4-10}{-3} = 2 \end{cases}$$

ES 2

Ricavare l'espressione per $\dfrac{Df}{Dt}$

$$F(X,t) = F\left[X_K^{-1}(x,t), t\right] = f(x,t)$$

$$\dot{f} = \left.\frac{\partial f(\chi_k(X,t),t)}{\partial t}\right|_X$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\dot{\chi}_{k,1} + \frac{\partial f}{\partial x_2}\dot{\chi}_{k,2} + \frac{\partial f}{\partial x_3}\dot{\chi}_{k,3}$$

$$= \frac{\partial f}{\partial t} + \sum_{j=1}^{3}\frac{\partial f}{\partial x_j}\dot{\chi}_{k,j} = \frac{\partial f}{\partial t} + u\cdot\nabla f = \frac{Df}{Dt}$$

ES 3

$$f(x,t) = x_1^2 + t\,x_2$$

$$u = a\,x_2\,\hat{e}_1 + b\,x_1\,\hat{e}_2$$

$$\nabla f = 2x_1\,\hat{e}_1 + t\,\hat{e}_2$$

$$\frac{Df}{Dt}(1,1) = \left(\frac{\partial f}{\partial t} + u\cdot\nabla f\right)_{(1,1)} = 1 + 2a + tb\,;$$

ES 4

La matrice Jacobiana della mappa di moto dell'esercizio 1 sarà in componenti

$$J_{ij} = \frac{\partial x_i}{\partial X_j} \Rightarrow J = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$$

$$\det J = |J| = 1 - t^2$$

$$|\dot{J}| = -2t$$

Il campo Euleriano invece

$$\begin{cases} u_1 = \dfrac{x_2 - t x_1}{1 - t^2} \\ u_2 = \dfrac{x_1 - t x_2}{1 - t^2} \end{cases}$$

$$\nabla \cdot u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = \frac{-2t}{1 - t^2}$$

$$\Rightarrow |\dot{J}| = |J| \, \nabla \cdot u = -2t$$

<u>ESS</u>

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla) u$$

In componenti

$$\frac{Du}{Dt} \cdot \hat{e}_1 = \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2}$$

$$\frac{Du}{Dt} \cdot \hat{e}_2 = \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2}$$

Nel punto $P = (1, 0)$ si ha:

$$\frac{\partial u_1}{\partial t} = -\frac{1+t^2}{(1-t^2)^2} \qquad \frac{\partial u_2}{\partial t} = \frac{2t}{(1-t^2)^2}$$

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = \frac{-t}{1-t^2}$$

$$\frac{\partial u_1}{\partial x_2} = \frac{1}{1-t^2} \qquad \frac{\partial u_2}{\partial x_1} = \frac{1}{1-t^2}$$

$$u_1 = \frac{-t}{1-t^2} \qquad u_2 = \frac{1}{1-t^2}$$

$$\frac{Du}{Dt} \cdot \hat{e}_1 = -\frac{1+t^2}{(1-t^2)^2} + \left(\frac{t}{1-t^2}\right)^2 + \frac{1}{(1-t^2)^2} = 0 \ \forall t$$

$$\frac{Du}{Dt} \cdot \hat{e}_2 = \frac{2t}{(1-t^2)^2} - \frac{t}{(1-t^2)^2} - \frac{t}{(1-t^2)^2} = 0 \ \forall t$$